Advanced Algorithms

March 5, 2026

Logistics

- Exercise set is posted, due next Thursday

- Office hours today after class

- Class attendance is part of your grade

- Course feedback:
  - Coding
  - Office hours timing
  - Level of difficulty
  - Posting my hand-written notes

- You can always submit more feedback, see course webpage

Duality

Hot Tubs Revisited

- You are running a Hot Tub production company.

- You can produce two types of hot tubs: Aqua-Spas and Hydro-Luxes.

- They require resources (pumps, labor, and tubing), and yield a certain profit

	Aqua-Spa	Hydro-Lux
Pumps	1	1
Labor	9 hours	6 hours
Tubing	12 feet	16 feet
Price	$350	$300

- You have 200 pumps, 1566 hours of labor, and 2880 feet of tubing.

- How many of each hot tub to produce if we want to maximize sales?

Linear Programming relaxation:

x_A = number of Aqua-Spas to produce

x_H = number of Hydro-Luxes to produce

Maximize: $350x_A + 300x_H$

Subject to:

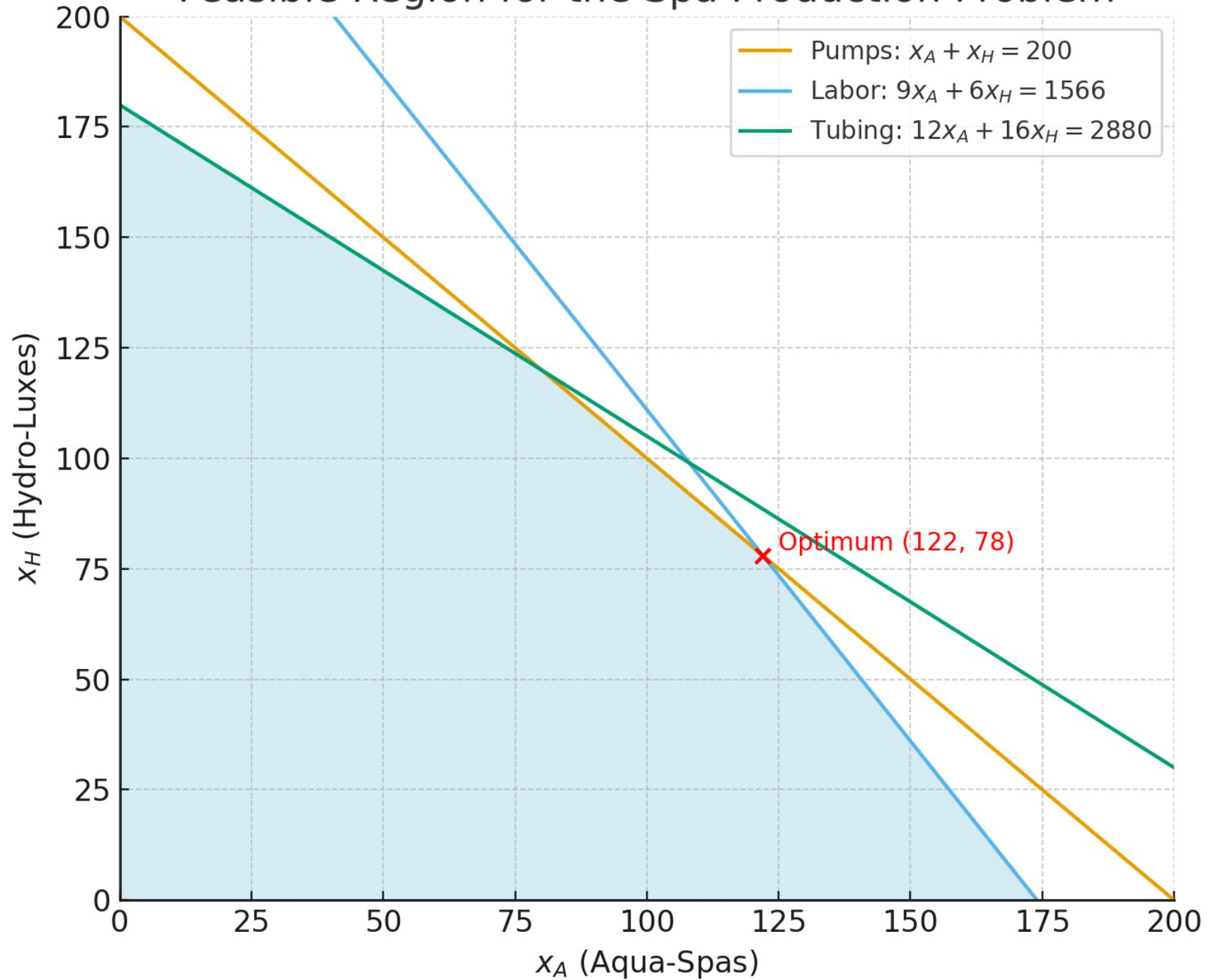$$x_A + x_H \leq 200 \qquad \text{(pumps)}$$
$$9x_A + 6x_H \leq 1566 \qquad \text{(labor)}$$
$$12x_A + 16x_H \leq 2880 \qquad \text{(tubing)}$$
$$x_A, x_H \geq 0 \qquad \text{(non-negativity)}$$

Optimal value: $66,100

Feasible Region for the Spa Production Problem

Pumps: $x_A + x_H = 200$
Labor: $9x_A + 6x_H = 1566$
Tubing: $12x_A + 16x_H = 2880$

Optimum (122, 78)

x_H (Hydro-Luxes)

x_A (Aqua-Spas)

Goal: upper bound the profit

Maximize: $350x_A + 300x_H$

Subject to:

$$x_A + x_H \leq 200 \qquad \text{(pumps)}$$
$$9x_A + 6x_H \leq 1566 \qquad \text{(labor)}$$
$$12x_A + 16x_H \leq 2880 \qquad \text{(tubing)}$$
$$x_A, x_H \geq 0 \qquad \text{(non-negativity)}$$

Upper Bounds

- Let's call the optimal value z^*

- Start with the pumps constraint :
  - $x_A + x_H \leq 200$

- Multiply both sides by 350:
  - $350x_A + 350x_H \leq 70{,}000$

- Observe upper bound:
  - $350x_A + 300x_H \leq 70{,}000$.   So $z^* \leq 70{,}000$.

- Note: we relied on the non-negativity constraints for this conclusion

Upper Bounds

- Try another constraint now

- Start with the labor constraint :
  - $9x_A + 6x_H \leq 1566$

- Multiply both sides by 50:
  - $450x_A + 300x_H \leq 78{,}300$

- Observe upper bound:
  - $350x_A + 300x_H \leq 78{,}300.$ So $z^* \leq 78{,}300$

- Note: we relied on the non-negativity constraints for this conclusion

Combining constraints

- Start with the pumps and labor constraints:
  - $x_A + x_H \leq 200$
  - $9x_A + 6x_H \leq 1566$

- Multiply pumps constraint by 300, labor constraint by 6:
  - $300x_A + 300x_H \leq 60{,}000$
  - $54x_A + 36x_H \leq 9{,}396$

- Add together
  - $354x_A + 336x_H \leq 69{,}396$

- Observe upper bound:
  - $350x_A + 300x_H \leq 69{,}396$.   So $z^* \leq 69{,}396$

General Strategy

- Start with the constraints:
  - $x_A + x_H \leq 200$
  - $9x_A + 6x_H \leq 1566$
  - $12x_A + 16x_H \leq 2880$

- Choose any three non-negative values y_1, y_2, y_3 and multiply:
  - $y_1(x_A + x_H) \leq y_1(200)$
  - $y_2(9x_A + 6x_H) \leq y_2(1566)$
  - $y_3(12x_A + 16x_H) \leq y_3(2880)$

- Check that the objective is "beaten"
  - $y_1 + 9y_2 + 12y_3 \geq 350$
  - $y_1 + 6y_2 + 16y_3 \geq 300$

- Observe upper bound:
  - $z^* \leq 200y_1 + 1566y_2 + 2880y_3$

General Strategy

- What's the best (lowest) upper bound we can get using this strategy?

- BIG idea: use a Linear Program.

- Decisions: y_1, y_2, y_3

- Objective: $\min\ 200y_1 + 1566y_2 + 2880y_3$

- Constraints:
  - $y_1 + 9y_2 + 12y_3 \geq 350$
  - $y_1 + 6y_2 + 16y_3 \geq 300$
  - $y_1, y_2, y_3 \geq 0$

This LP is the dual of the original LP

Your turn:

Consider the following LP:

$$\max 2x_1 + 7x_2 + 4x_3$$

$$x_1 + 2x_2 + x_3 \leq 10$$
$$3x_1 + 3x_2 + 2x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Show that the optimal value cannot exceed 25

Duality

Primal:

$$\max \quad 350x_A + 300x_H$$
$$s.t. \quad x_A + x_H \leq 200$$
$$9x_A + 6x_H \leq 1566$$
$$12x_A + 16x_H \leq 2880$$
$$x_A, x_H \geq 0$$

Dual:

$$\min \quad 200y_1 + 1566y_2 + 2880y_3$$
$$s.t. \quad y_1 + 9y_2 + 12y_3 \geq 350$$
$$y_1 + 6y_2 + 16y_3 \geq 300$$
$$y_1, y_2, y_3 \geq 0$$

By construction, every feasible solution to the dual yields an upper bound on the value of the primal

primal solutions   dual solutions
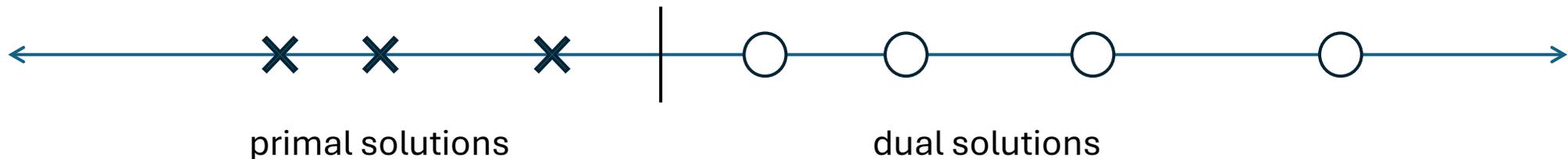
Duality

All Linear Programs come in dual pairs. One maximizes one minimizes.
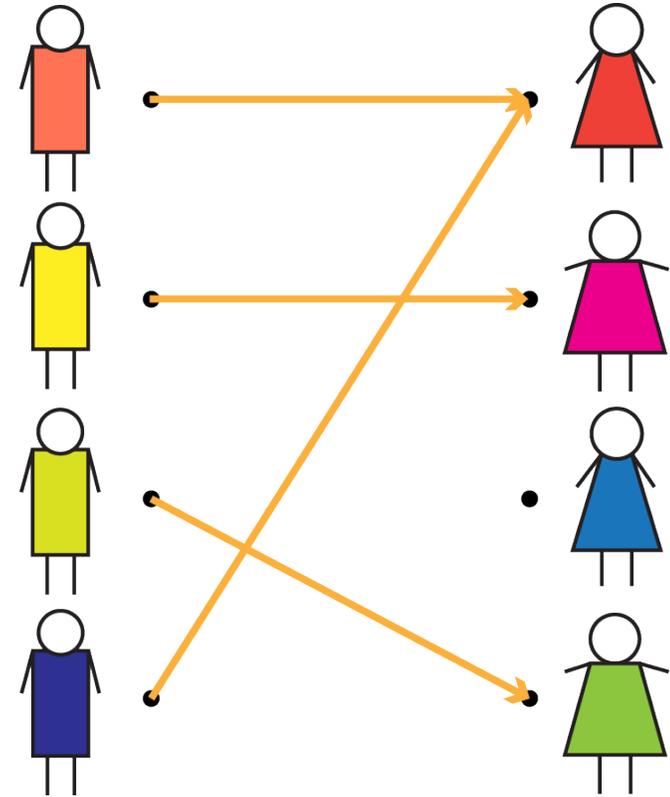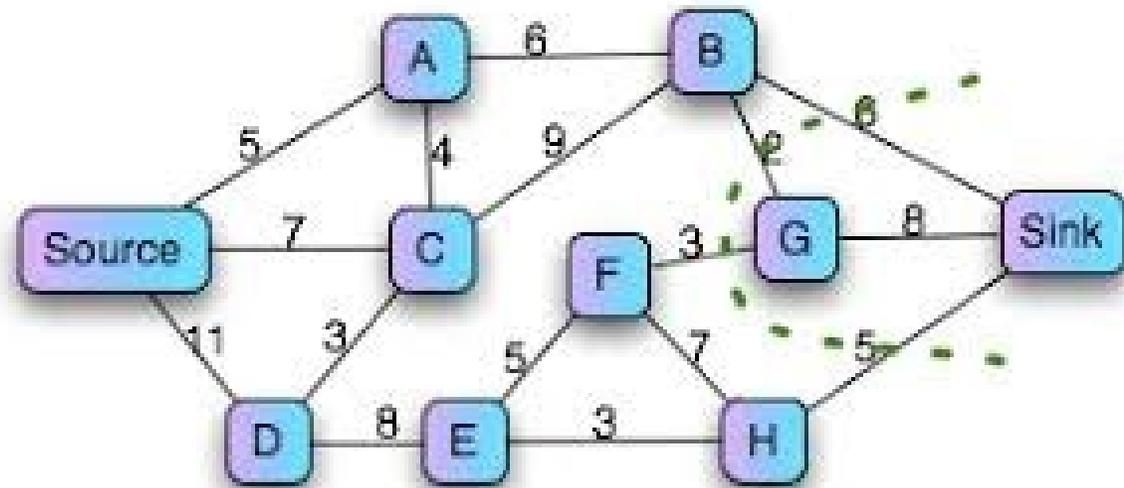
Weak Duality: given a primal LP and its dual LP:

objective value of any feasible solution to the maximization problem $\leq$ objective value of any feasible solution to the minimization problem

Strong Duality: if primal has an optimal solution, then so does its dual, and their optimal values are equal.

OPT(primal) = OPT(dual)
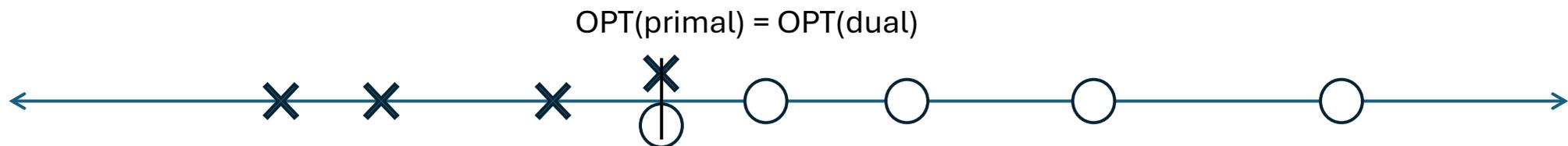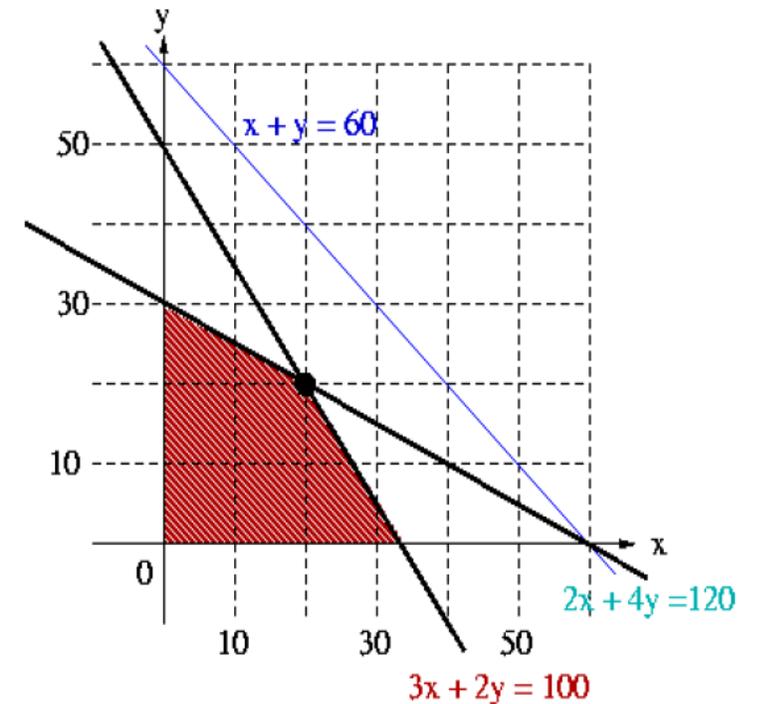
Duality in Algorithm Design

Complementary Slackness

Recall: a constraint can be either tight or slack at the optimal solution

At an optimal solution:

- if a variable is **non-zero**, then its corresponding dual constraint is **tight**

(Equivalently, if a constraint is **slack,** the corresponding dual variable is **zero)**

The two binding constraints at the optimum

Hot Tubs

Primal:

$$\max \quad 350x + 300x_H$$
$$s.t. \quad x_A + x_H \leq 200$$
$$9x_A + 6x_H \leq 1566$$
$$12x_A + 16x_H \leq 2880$$
$$x_A, x_H \geq 0$$

Dual:

$$\min \quad 200y_1 + 1566y_2 + 2880y_3$$
$$s.t. \quad y_1 + 9y_2 + 12y_3 \geq 350$$
$$y_1 + 6y_2 + 16y_3 \geq 300$$
$$y_1, y_2, y_3 \geq 0$$

Your turn:

The optimal solution to the primal LP is $(x_A, x_H) = (122, 78)$.

Use complementary slackness to find the optimal solution to the dual.